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QUALITATIVE PROPERTIES OF SOLUTIONS OF VOLTERRA EQUATIONS IN BANACH SPACES

BY
PHILIPPE CLÉMENT AND ENZO MITIDIERI
Department of Mathematics and Informatics, Delft University of Technology, 2600 AJ-Delft, The Netherlands; and Dipartimento di Scienze Matematiche, Università degli Studi di Trieste, 34100 Trieste, Italy

ABSTRACT
In this paper we consider an abstract Volterra integral equation in an ordered Banach space. We establish some monotonicity properties of solutions and apply these results to their asymptotic behaviour. This is done by using the method of upper and lower solutions.

1. Introduction

The purpose of this paper is to study some qualitative properties of solutions of the abstract integral equation

$$(1.1) \qquad u(t)+\int_0^t b(t-s)Au(s)ds \ni u_0+\int_0^t b(t-s)g(s)ds, \qquad t \geqq 0$$

in a real Banach space X.

Here A is an m-accretive operator, possibly multivalued in X, $u_0 \in \overline{D(A)}$, the closure of the domain of A, b is a real kernel and $g \in L^1_{\mathrm{loc}}(0,+\infty;X)$.

Throughout this paper we shall consider equation (1.1) for a class of kernels which was introduced in [7] (see H_4 and H_5) and called completely positive in [8] (see Section 3).

The existence of solutions in a suitable sense was considered in [9], [13] and more recently in [14].

We shall frequently refer to these papers for the existence theory concerning

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(1.1). The choice of the class of complete positive kernels is not only important for the development of reasonable existence theory, but is also of great influence in the consideration of qualitative properties of solutions.

In fact, recent studies on the positivity of solutions of (1.1) when X is an ordered Banach space (see [3], [8]) or on the asymptotic behavior of solutions as t goes to infinity (see [4], [5], [6], [7], [8], [15], [16], [17], [19], [21], [22]) show that the complete positivity of b plays a crucial role in the analysis.

In this paper we shall discuss, under some suitable assumptions, special properties enjoyed by the solutions to (1.1), namely, invariance with respect to a closed convex set contained in X, monotonicity (decreasingness or increasingness) and asymptotic behavior as t goes to infinity.

Some results of this paper complement preceding work of Friedman [10]–[12] about monotonicity of solutions.

The paper is organized as follows: Section 2 contains some notations and preliminary facts concerning (1.1). In Section 3 we give a new characterization of the class of complete positive kernels and we correct a gap in the proof of Theorem 2.2 of [8], p. 520 (2.15) which will be used in an essential way throughout the rest of the paper. Section 4 is mainly devoted to the proof of a "singular perturbation" result (see Proposition 4.1) which will be used in Section 5 and seems to be of interest in itself.

Finally Section 5 contains a general invariance result of solutions to (1.1) with respect to a closed convex set contained in X and a sufficient condition (see Theorem 5.3) which insures, in the case when X is an ordered Banach space, that the solutions of (1.1) are decreasing or increasing in time.

We conclude the paper with an application to the study of the asymptotic behavior of the solutions to a class of semilinear Volterra equations in an ordered Banach space extending to the Volterra equations the classical results of Sattinger (see [23]).

2. Preliminaries

Throughout this paper we shall use the following notations. X is a real Banach space endowed with the norm $\|\cdot\|$. The operator $A : D(A) \subset X \to 2^X$ is m-accretive (see [9]), and when A is linear we shall assume as usual that A is single-valued.

We denote the resolvent and the Yosida approximation of A by J_λ and A_λ, that is

$$J_\lambda := (1 + \lambda A)^{-1}, \qquad A_\lambda := \lambda^{-1}(1 - J_\lambda) \qquad \text{for every } \lambda > 0.$$

Given $T>0$ and a Banach space Y, we use the standard notation $L^1(0, T; Y)$ for the space of Bochner integrable (equivalence classes of) Y-valued functions.

If $Y=R$, we write $L^1(0, T)$ instead of $L^1(0, T; R)$. If $b\in L^1(0, T)$ and $f\in L^1(0, T; Y)$ we define

$$(b*f)(t):=\int_0^t b(t-s)f(s)ds, \qquad t\in[0, T].$$

Let us recall that if $f\in L^p(0, T; Y)$ for some $p\in[1,+\infty]$, then $b*f\in L^p(0, T; Y)$ and

$$\| b*f\|_p \leqq \| b\|_1 \| f\|_p \tag{2.1}$$

where $\| b\|_1:=\int_0^T|b(s)|ds$ and

$$\| f\|_p:=\left(\int_0^T \| f(s)\|^p ds\right)^{1/p} \qquad \text{if } 1\leqq p<\infty$$

and

$$\| f\|_\infty:=\operatorname*{ess\,sup}_{t\in[0,T]} \| f(t)\| \qquad \text{if } p=\infty.$$

For $f\in L^1(0, T; Y)$ and $b\in L^1(0, T)$ the equation

$$u+b*u=f \qquad \text{on } [0, T] \tag{2.2}$$

possesses a unique solution u in $L^1(0, T; Y)$.

In particular if $Y=R$ and $f=b$ (resp. $f=1$) we denote the solution to (2.2) by $r(b)$ (resp. by $s(b)$).

Then the solution to (2.2) is given by

$$u=f-r(b)*f. \tag{2.3}$$

If $f=b*g$, for some $g\in L^1(0, T; Y)$ then $u=r(b)*g$, hence the unique solution of

$$u+b*u=u_0+b*g \qquad \text{on } [0, T] \tag{2.4}$$

with $u_0\in Y$ and $g\in L^1(0, T; Y)$ is given by

$$u(t)=s(b)(t)u_0+(r(b)*g)(t), \qquad t\in[0, T]. \tag{2.5}$$

From (2.2) we also have

$$s(b)(t)=1-(1*r(b))(t)=1-\int_0^t r(b)(s)ds \tag{2.6}$$

which implies that $s(b) \in AC(0, T)$ and

$$r(b)(t) = -s(b)'(t) \qquad \text{a.e. on } [0, T].$$

Finally consider the equation

$$u + b * A_\lambda u = u_0 + b * g \tag{2.7}$$

with $b \in L^1(0, T)$, $u_0 \in X$, $g \in L^1(0, T; X)$ and $\lambda > 0$.

From the definition of A_λ and (2.4), it follows that (2.7) is equivalent to

$$u(t) = (r(\lambda^{-1}b) * (J_\lambda u + \lambda g))(t) + s(\lambda^{-1}b)(t)u_0, \qquad t \in [0, T]. \tag{2.8}$$

Using a standard contraction argument in $L^1(0, T; X)$ one proves the existence and uniqueness of a solution u_λ of (2.7) in $L^1(0, T; X)$.

Moreover

$$u_\lambda = \lim_{n \to \infty} u_\lambda^{(n)} \qquad \text{in } L^1(0, T; X)$$

where

$$\begin{aligned} u_\lambda^{(0)} &= \text{arbitrary element of } L^1(0, T; X), \\ u_\lambda^{(n)} &= r(\lambda^{-1}b) * (J_\lambda u_\lambda^{(n-1)} + \lambda g) + s(\lambda^{-1}b)u_0. \end{aligned} \tag{2.9}$$

Note that if $T^* \in (0, T)$ then the restriction on $[0, T^*]$ of a solution of (2.7) on $[0, T]$ is the solution of (2.7) on $[0, T^*]$.

Finally we shall use the notations ($1 \leqq p \leqq \infty$)

$$W_0^{1,p}(0, T; Y) := \{u \in W^{1,p}(0, T; Y) : u(0) \equiv 0\}$$

and the standard abbreviations BV — bounded variation, AC — absolutely continuous, C — continuous, and the corresponding functional spaces.

3. Completely positive kernels

In this section we shall consider three equivalent characterizations of a class of kernels which were introduced in [7].

THEOREM 3.1. *Let $b \in L^1(0, T)$, for some $T > 0$. Then the following assertions are equivalent:*

(i) *$s(\lambda b)(\cdot)$ is nonnegative and nonincreasing on $[0, T]$ for every $\lambda > 0$, and b is not identically zero.*

(ii) *There exist $\alpha \geqq 0$ and $k \in L^1(0, T)$ nonnegative and nonincreasing such that*

$$\alpha b(t) + (k * b)(t) = 1 \qquad on\ [0, T]. \tag{3.1}$$

(iii) *Let $B_p: L^p(0, T) \to L^p(0, T)$, $p \in [1, +\infty]$ be defined by*

$$B_p u := b * u \qquad for\ u \in L^p(0, T). \tag{3.2}$$

Then B_p is injective and $\Gamma_p := B_p^{-1}$ is m-accretive in $L^p(0, T)$, for all $p \in [1, +\infty]$, with nonnegative resolvent $J_\lambda = (1 + \lambda\Gamma_p)^{-1}$ for every $\lambda > 0$.

In order to prove Theorem 3.1 we need the following:

LEMMA 3.2. *Let $b \in L^1(0, T)$ be such that there exist $\alpha \in R$ and $k \in L^1(0, T)$ satisfying* (3.1). *Define B_p as in* (3.2) *for $p \in [1, +\infty]$. Then*:

(i) *B_p is injective.*

(ii) *Γ_p, the inverse of B_p, satisfies*

$$D(\Gamma_p) = \{u \in L^p(0, T) : \alpha u + k * u \in W_0^{1,p}(0, T)\},$$

$$\Gamma_p u = (\alpha u + k * u)'.$$

(iii) *For every $\lambda > 0$,*

$$I + \lambda\Gamma_p : D(\Gamma_p) \to L^p(0, T)$$

is a bijection and

$$(1 + \lambda\Gamma_p)^{-1} u = r(\lambda^{-1} b) * u.$$

PROOF. (i) Assume $u \in L^p(0, T)$, for some $p \in [1, +\infty]$ and $b * u = 0$. Then (3.1) implies

$$1 * u = \alpha b * u + k * b * u = 0$$

on $[0, T]$, hence $u = 0$ a.e. on $[0, T]$.

(ii) First we prove $\Gamma_p \cdot B_p = I$. Let $u \in L^p(0, T)$, then

$$\alpha b * u + k * b * u = 1 * u \in W_0^{1,p}(0, T).$$

Thus $B_p u \in D(\Gamma_p)$ and $\Gamma_p \cdot B_p = (1 * u)' = u$.

Next we prove $B_p \cdot \Gamma_p = I_{D(\Gamma_p)}$. Let $u \in D(\Gamma_p)$, then

$$b * (\alpha u + k * u)' = [(\alpha u + k * u) * b]' = (1 * u)'$$

by using that $\alpha u + k * u \in W_0^{1,p}(0, T)$.

(iii) Let $\lambda > 0$. We have for $f \in L^p(0, T)$ and $u \in D(\Gamma_p)$:

$$(I + \lambda\Gamma_p)u = f \Leftrightarrow B_p u + \lambda u = B_p f$$

$$\Leftrightarrow u + \lambda^{-1} b * u = \lambda^{-1} b * f \Leftrightarrow u = r(\lambda^{-1} b) * f.$$

This completes the proof of Lemma 3.2.

REMARK 3.3. Note that if the kernel b of Theorem 3.1 satisfies (iii) then it also satisfies (iii) with $L^p(0, T)$ replaced by $L^p(0, T; X)$.

PROOF OF THEOREM 3.1. (i)⇒(ii). Let $b \in L^1(0, T)$ satisfy (i). For $\lambda > 0$, set $v_\lambda = \lambda^{-1} r(\lambda b)$. Then v_λ satisfies

$$v_\lambda + \lambda b * v_\lambda = b \qquad \text{on } [0, T].$$

By using (2.1) we obtain

$$\| v_\lambda \|_1 \leqq \| b \|_1 (1 + \lambda \| v_\lambda \|_1)$$

which implies that $\| v_\lambda \|_1$ is bounded as a function of λ for $\lambda \in (0, \frac{1}{2} \| b \|_1^{-1}]$. Consequently

$$\lim_{\lambda \downarrow 0} \| v_\lambda - b \|_1 = 0,$$

and b is nonnegative since $v_\lambda = \lambda^{-1} r(\lambda b)$ is nonnegative. Since b is not identically zero on $[0, T]$, $\int_0^T b(s)ds > 0$.

As in [8], p. 519 we obtain the first estimate

$$\sup_{\lambda > 0} \int_0^T \lambda s(\lambda b)(\gamma) d\gamma \leqq 2T \left(\int_0^T b(\gamma) d\gamma \right)^{-1}. \tag{3.3}$$

Indeed define for $\lambda > 0$,

$$\tilde{s}(\lambda b)(t) = \begin{cases} s(\lambda b)(t), & t \in [0, T], \\ 0, & t \in (T, +\infty), \end{cases}$$

and

$$\tilde{b}(t) = \begin{cases} b(t), & t \in [0, T], \\ 0, & t \in (T, +\infty). \end{cases}$$

If $z_\lambda(t) = \tilde{s}(\lambda b)(t) + (\lambda \tilde{b} * s(\lambda \tilde{b}))(t)$ for $t \geqq 0$ and $\lambda > 0$, then it is easily verified that $z_\lambda(t) = 1$ for $t \in [0, T]$, $z_\lambda(t) = 0$ for $t > 2T$.

For $t \in (T, 2T)$, using the fact that $\tilde{s}(\lambda b)(\cdot)$ is nonnegative, one gets

$$(\tilde{s}(\lambda b) + \lambda \tilde{b})(t) = \int_0^T \tilde{s}(\lambda b)(t - \gamma) \lambda \tilde{b}(\gamma) d\gamma \leqq \int_0^T \tilde{s}(\lambda b)(t - \gamma) \lambda b(\gamma) d\gamma \leqq 1.$$

Thus $z_\lambda(t) \in [0, 1]$, for $t \in [T, 2T]$ and $\lambda > 0$, hence from the convolution theorem we obtain

$$\int_0^\infty \tilde{s}(\lambda b)(\gamma)d\gamma\left(1 + \int_0^\infty \lambda \tilde{b}(\gamma)d\gamma\right) = \int_0^\infty z_\lambda(\gamma)d\gamma$$

and

$$\left(\lambda \int_0^T s(\lambda b)(\gamma)d\gamma\right)\left(\int_0^T b(\gamma)d\gamma\right) \leqq 2T$$

which implies (3.3).

Next we define

$$v_\lambda(t) = \int_t^T \lambda s(\lambda b)(\gamma)d\gamma, \qquad t \in [0, T]. \tag{3.4}$$

Since $\lambda s(\lambda b)(\cdot)$ is nonnegative,

$$\mathrm{Var}[v_\lambda; [0, T]] = \int_0^T \lambda s(\lambda b)(\gamma)d\gamma.$$

and from (3.3)

$$\operatorname*{Sup}_{\lambda > 0} \mathrm{Var}[v_\lambda; [0, T]] < +\infty.$$

Note that $v_\lambda(T) = 0$, hence v_λ are uniformly bounded on $[0, T]$. Hence, from Helly's theorem, there is $v: [0, T] \to R$, nonincreasing, and convex satisfying

$$\mathrm{Var}[v: [0, T]] \leqq 2T\left(\int_0^T b(\gamma)d\gamma\right)^{-1}$$

and there exists a sequence $\lambda_n \uparrow \infty$ as $n \to \infty$ such that

$$\lim_{n\to\infty} \int_0^T g(t)dv_{\lambda_n}(t) = \int_0^T g(t)dv(t)$$

holds for every $g \in C[0, T]$.

Next, for every $u \in C[0, T]$, one has

$$s(\lambda_n b) * u + \lambda_n b * s(\lambda_n b) * u = 1 * u. \tag{3.5}$$

From (3.4), we have $v'_{\lambda_n}(t) = -\lambda_n s(\lambda_n b)(t)$ and (3.5) becomes

$$\begin{aligned}(3.6)\qquad &\lambda_n^{-1}\int_0^t u(t-\gamma)dv_{\lambda_n}(\gamma)+\int_0^t (u*b)(t-\gamma)dv_{\lambda_n}(\gamma)\\ &=-\int_0^t u(\gamma)d\gamma,\qquad t\in[0,T].\end{aligned}$$

By taking the limit as $n\to\infty$, we obtain

$$(3.7)\qquad \int_0^t (u*b)(t-\gamma)dv(\gamma)=-\int_0^t u(\gamma)d\gamma,\qquad t\in[0,T].$$

From (3.7), we infer that if $b*u=0$ on $[0,T]$, then $\int_0^t u(\gamma)d\gamma=0$ on $[0,T]$ hence $u=0$ on $[0,T]$.

We claim that $\int_0^t b(\gamma)d\gamma>0$ for $t\in(0,T]$.

Otherwise, there exists $t_0\in(0,T]$ such that

$$\int_0^{t_0} b(\gamma)d\gamma=0.$$

and since $b(\cdot)$ is nonnegative, $b(t)=0$ a.e. on $[0,t_0]$. But then we can choose $\varepsilon\in(0,t_0)$ such that $b(t)=0$ on $[0,\varepsilon]$ and $u\in C[0,T]$ satisfying

$$\begin{cases}u(t)=0 & \text{for } t\in[0,T-\tfrac12\varepsilon]\\ u(t)>0 & \text{for } t\in(T-\tfrac12\varepsilon,T].\end{cases}$$

Then $(b*u)=0$ for $t\in[0,T]$ while $u\neq 0$ contradicting (3.7). Thus $\int_0^t b(\gamma)d\gamma>0$ for $t\in(0,T]$.

Since $s(\lambda b)(t)$ is nonnegative, and nonincreasing, we obtain

$$(3.8)\qquad \lambda s(\lambda b)(t)\leqq\left(\int_0^t b(\gamma)d\gamma\right)^{-1}\qquad \text{for } t>0.$$

This implies that, for every $\varepsilon\in(0,T)$, $v\in\mathrm{Lip}(\varepsilon,T)$ and *a fortiori* $v\in\mathrm{AC}[\varepsilon,T]$.

Define

$$e(t)=\begin{cases}0, & t=0,\\ 1, & t\in(0,T],\end{cases}$$

and

$$h(t):=v(t)+(v(0)-v(0^+))e(t).$$

Then $h(\cdot)$ is continuous, nonincreasing and convex on $[0,T]$. Moreover

$$h(t)=-\int_t^T h'(\gamma)d\gamma,\qquad t\in(0,T].$$

We obtain that $h \in AC[0, T]$. Therefore $v(t) = -\alpha e(t) + h(t)$, with $\alpha \geqq 0$, $h \in AC[0, T]$ nonincreasing and convex. By using (3.7) we obtain

$$-\alpha(u * b)(t) + (u * b * h')(t) = -(1 * u)(t). \tag{3.9}$$

If we denote $-h'$ by $k \in L^1(0, T)$, then $k(\cdot)$ is nonnegative and nonincreasing, and

$$(\alpha b * u)(t) + ((k * b) * u)(t) = (1 * u)(t) \qquad \text{on } [0, T]$$

for every $u \in C(0, T)$.

It follows that

$$\alpha b + k * b = 1 \qquad \text{on } (0, T].$$

This completes the proof of the implication (i)$\Rightarrow$(ii).

For the implication (ii)$\Rightarrow$(i) see [8].

Next we prove (ii)$\Rightarrow$(iii). By using (ii) and Lemma (3.2)(ii), we have that B_p is injective with inverse Γ_p and $I + \lambda\Gamma_p$ is a bijection from $D(\Gamma_p)$ onto $L^p(0, T)$ with

$$(I + \lambda\Gamma_p)^{-1}u = r(\lambda^{-1}b) * u \qquad \text{for every } u \in L^p(0, T) \quad \text{and} \quad \lambda > 0.$$

By (i) we know that $r(\lambda^{-1}b) \geqq 0$ and from (2.6) and $s(\lambda^{-1}b) \geqq 0$ it follows that

$$\| r(\lambda^{-1}b) \|_1 \leqq 1.$$

By using (2.1), we obtain

$$\| (I + \lambda\Gamma_p)^{-1}u \|_p \leqq \| u \|_p$$

and $(I + \lambda\Gamma_p)^{-1}u \geqq 0$ whenever $u \geqq 0$ a.e. on $[0, T]$.

Finally we prove (iii)$\Rightarrow$(i). If b satisfies (iii), then for each $f \in L^1(0, T)$, $f \geqq 0$ a.e. on $[0, T]$, the equation

$$u + \lambda\Gamma_1 u = f, \qquad \lambda > 0,$$

possesses a unique solution $u \in D(\Gamma_1)$, satisfying

$$u_\lambda \geqq 0 \qquad \text{on } [0, T] \tag{3.10}$$

and

$$\| u_\lambda \|_1 \leqq \| f \|_1.$$

Then u_λ is also the solution to

$$B_1 u_\lambda + \lambda u_\lambda = B_1 f$$

or equivalently to

$$u_\lambda + \lambda^{-1} b * u = \lambda^{-1} b * f,$$

that is $u_\lambda = r(\lambda^{-1} b) * f$.

Then (3.10) implies $r(\lambda^{-1}b)(\cdot) \geqq 0$ on $[0, T]$ and $\| r(\lambda^{-1}b) \|_1 \leqq 1$. Hence

$$s(\lambda^{-1}b)(t) = 1 - \int_0^t r(\lambda^{-1}b)(\gamma) d\gamma \geqq 0.$$

This completes the proof of Theorem 3.1.

Remark 3.4. (i) Given b completely positive on $[0, T]$, then α and $k(\cdot)$ of Theorem 3.1(ii) are uniquely defined. We first consider the case where $b \in L^\infty(0, T)$. We have $b \in L^\infty(0, T)$ iff $\alpha > 0$.

If $b \in L^\infty(0, T)$ then

$$\lim_{t \downarrow 0} (k * b)(t) = 0$$

and thus

$$\lim_{t \downarrow 0} \alpha b(t) = 1,$$

hence $\alpha > 0$.

If $\alpha > 0$, then $b = \alpha^{-1} s(\lambda^{-1} k) \in \mathrm{AC}[0, T]$. we deduce, by differentiating (3.1),

$$b(0)k + b' * k = -\alpha b' \tag{3.11}$$

hence $\alpha = b(0)^{-1}$ and

$$k = -\alpha r(\alpha b');$$

thus if $b \in L^\infty(0, T)$, α and $k(\cdot)$ are uniquely defined.

If $b \notin L^\infty(0, T)$ then there is $k \in L^1(0, T)$, nonnegative and nonincreasing such tht $k * b = 1$. The homogeneous equation $b * u = 0$ with $b \in L^1(0, T)$ possesses only the trivial solution $u = 0$, since $b * u = 0$ implies $k * b * u = 0$, hence $1 * u = 0$. Thus k is uniquely defined.

(ii) Note that if $\alpha > 0$, then $b \in \mathrm{AC}[0, T]$, hence $r(\lambda b) \in \mathrm{AC}[0, T]$, and $s(\lambda b) \in W^{2,1}(0, 1)$ for every $\lambda > 0$.

Remark 3.5. We recall that if $b \in L^1(0, T)$ is positive nonincreasing and $\log b$ is (in particular if b is completely monotonic), then b is completely positive.

(iii) It follows from Theorem 3.1 that

$$\int_0^T \Gamma_2 u(s)u(s)ds \geqq 0 \qquad \text{for every } u \in D(\Gamma_2)$$

hence

$$\int_0^T B_2 u(s)u(s)ds \geqq 0 \qquad \text{for every } u \in L^2(0, T).$$

In particular, if b is completely positive on $[0, T]$ for every $T > 0$, then b is of positive type (see [20]).

Finally, we observe that $b(t) = \exp(-t^2)$ is positive and of positive type but not completely positive.

Indeed if $b \in \mathrm{AC}[0, T]$ and $b(0) \neq 0$, then $\alpha b + k * b = 1$ with $\alpha^{-1} = b(0)$ and $k = -\alpha r(\alpha b')$.

Then *if* $b(t) = \exp(-t^2)$, $b'(0) = 0$ and $k(0) = 0$.

Thus k cannot be positive and decreasing.

REMARK 3.6. If b is completely positive ($b \neq 0$) on $[0, T]$, then for every $\lambda > 0$ and every $t \in (0, T]$ we have $s(\lambda^{-1}b) < 1$.

Otherwise, since $s(\lambda^{-1}b)(\cdot)$ is nonincreasing for every $\lambda > 0$, there would be a $t_0 \in (0, T]$ such that $s(\lambda^{-1}b)(t) = 1$ for $t \in (0, t_0]$, hence $r(\lambda b)(t) = 0$ on $[0, t_0]$ and $b(t) = 0$ on $[0, t_0]$, but this would imply $0 \leqq b(t) < b(0) = 0$ for every $t \in [0, T]$, a contradiction.

Finally we prove the following:

PROPOSITION 3.7. *Let b be completely positive on* $[0, T]$ *and let* Γ_1 *be the operator defined in Theorem* 3.1 ($p = 1$), *with* $L^1(0, T)$ *replaced by* $L^1(0, T; X)$.

Then $-\Gamma_1$ *is the infinitesimal generator of a* C^0*-contraction semigroup of type* $\omega_0 = -\infty$.

PROOF. By Lemma 3.2, $D(\Gamma_1) \supset W_0^{1,1}(0, T, X)$ and $D(\Gamma_1)$ is dense in $L^1(0, T; X)$.

Since Γ_1 is m-accretive in $L^1(0, T; X)$ it follows that $-\Gamma_1$ is the infinitesimal generator of a C^0-contraction semigroup in $L^1(0, T; X)$.

Concerning the type we first consider the case when $X = R$ and the complexification of Γ_1 in $\mathbf{C}$.

For every $\lambda \in \mathbf{C}$, the equation

$$\lambda u + \Gamma_1 u = f \tag{3.12}$$

where $u, f \in L^1(0, T, \mathbf{C})$ is equivalent to

$$u + \lambda b * u = b * u, \tag{3.13}$$

by noting that b is real valued and using Lemma 3.2.

But it is well known that for every $f \in L^1(0, T; \mathbf{C})$, (3.13) possesses a unique solution $u \in L^1(0, T; \mathbf{C})$.

Therefore the resolvent set of Γ_1 is empty.

Since the solution of (3.12) for $f \in L^1(0, T; R)$, $f \geqq 0$ and $\lambda \in R - 0$ is given by

$$u = \lambda^{-1} r(\lambda b) * b$$

and

$$u = b * f \qquad \text{for } \lambda = 0.$$

and is nonnegative, $-\Gamma_1$ generates a positive C^0-semigroup on $L^1(0, T)$.

Then it follows from [2, Cor. 1.3, p. 294] that the type of $-\Gamma_1$ is $-\infty$.

Next we consider the case where X is a real Banach space. We shall denote by Γ_1 (respectively S_{Γ_1}) the operator defined in Theorem 3.1 (respectively the C^0-semigroup generated by $-\Gamma_1$).

We shall denote by Γ the operator Γ_1 when $X = R$, and by $S(\cdot)$ the C^0-semigroup generated by $-\Gamma$ in $L^1(0, T)$.

From the exponential formula we have

$$S_{\Gamma_1}(t)u = \lim_{n\to\infty} (I + tn^{-1}\Gamma_1)^{-n}u, \qquad u \in L^1(0, T; X) \quad \text{and} \quad t \geqq 0 \tag{3.14}$$

and

$$S_\Gamma(t)v = \lim_{n\to\infty} (I + tn^{-1}\Gamma)^{-n}v, \qquad v \in L^1(0, T) \quad \text{and} \quad t \geqq 0. \tag{3.15}$$

For $u \in L^1(0, T, X)$ and $t > 0$ $(n \in N)$, we have, by using Lemma 3.2 and the fact that $r(\lambda) := r(\lambda b)$ is nonnegative,

$$\begin{aligned} \| (I + tn^{-1}\Gamma_1)^{-n}u \|_{L^1(0,T;X)} &\leqq \| r(nt^{-1}) * \cdots * r(nt^{-1}) * u \|_{L^1(0,T,X)} \\ &\leqq \| r(nt^{-1}) * \cdots * r(nt^{-1}) * \| u(\cdot) \| \|_{L^1(0,T)} \\ &\leqq \| (I + tn^{-1}\Gamma)^{-n} \| u(\cdot) \| \|_{L^1(0,T)}. \end{aligned} \tag{3.16}$$

Note that if $u \in L^1(0, T; X)$ and $\| u \|_{L^1(0,T,X)} \leqq 1$, then $\| u(\cdot) \| \in L^1(0, T)$ and $\| \; \| u(\cdot) \| \; \|_{L^1(0,T)} \leqq 1$.

By taking the limit as $n \to \infty$ in (3.16) we obtain

$$\| S_{\Gamma_1}(t)u \|_{L^1(0,T;X)} \leqq \| S(t) \| u(\cdot) \| \quad \|_{L^1(0,T)}, \qquad t \geqq 0$$

hence

$$\| S_{\Gamma_1}(t) \|_{L^1(0,T;X)} \leqq \| S(t) \|_{L^1(0,T)}, \qquad t \geqq 0. \tag{3.17}$$

(It is easily verified that equality holds in (3.17).)

Then

$$\omega_0 = \lim_{t \to \infty} \frac{1}{t} \log \| S_{\Gamma_1}(t) \| = \lim_{t \to \infty} \frac{1}{t} \log \| S_\Gamma(t) \| = -\infty.$$

This completes the proof of Proposition 3.6.

4. Existence and regularity results

In this section we recall some existence and regularity results for equation (1.1) obtained by Gripenberg (see [14]) when b is completely positive.

Let $u_0 \in X$ and $g \in L^1(0, T; X)$. By using Theorem 3.1 we can rewrite (2.7) in the following way:

$$\begin{aligned} &(\alpha u + k * u)' + A_\lambda u = k(t) u_0 + g(t), \\ &u(0) = u_0. \end{aligned} \tag{4.1}$$

Indeed if $u \in L^1(0, T; X)$ satisfies (2.7) then we have

$$\alpha u + \alpha b * A_\lambda u = \alpha u_0 + \alpha b * g$$

and

$$k * u + k * b * A_\lambda u = k * u_0 + k * b * g$$

where α and $k(\cdot)$ are defined in Theorem 3.1.

By adding we obtain

$$\alpha u + k * u + 1 * A_\lambda u = \alpha u_0 + (k * 1) u_0 + 1 * g, \tag{4.2}$$

hence

$$\alpha u + k * u \in W_0^{1,1}(0, T; X) \tag{4.3}$$

and u satisfies (4.1).

Conversely if

$$\alpha u = k * u \in W_0^{1,1}(0, T, X)$$

and satisfies (4.1), we have

$$b * (\alpha u + k * u)' + b * A_\lambda u = b * u_0 + b * g,$$

and since

$$
\begin{aligned}
u = (1 * u)' &= (\alpha b + k * b) * u \\
&= (\alpha u + k * u) * b \\
&= \alpha u(0) b + b * (\alpha u + k * u)'
\end{aligned}
$$

we have

$$u + b * A_\lambda u = (\alpha b + k * b) u_0 + b * g = u_0 + b * g.$$

Hence u satisfies (2.7).

It has been proved by Gripenberg (see [14]) that if $\alpha \geqq 0$, $k \in L^1(0, T)$ is positive and nonincreasing and $u_0 \in D(A)$, then u_λ (the unique solution to (2.7)) converges to some $u \in L^1(0, T; X)$ as λ goes to zero.

We shall call u the generalized solution of (1.1). Note that if $u \in L^1(0, T; X)$ satisfies (2.7), then u is the generalized solution of (2.7).

Moreover, if X is reflexive there exists $w \in L^1(0, T; X)$ such that

$$u + b * w = u_0 + b * g \qquad \text{a.e. on } [0, T] \tag{4.4}$$

and $u(t) \in D(A)$ a.e. on $[0, T]$, and $w(t) \in Au(t)$ a.e. on $[0, T]$. Hence it follows that $\alpha u + k * u \in W^{1,1}(0, T, X)$ and u satisfies

$$
\begin{aligned}
&(\alpha u + k * u)' + w = k(t) u_0 + g(t) \qquad \text{a.e. on } [0, T], \\
&u(0) = u_0.
\end{aligned} \tag{4.5}
$$

We recall also that if X is reflexive, $\alpha > 0$, $b' \in \mathrm{BV}(0, T)$ and $g \in \mathrm{BV}(0, T; X)$, then $w \in L^\infty(0, T; X)$ and $u \in W^{1,1}(0, T; X)$.

Moreover the following estimates hold for generalized solutions (see [5]):

$$\| u_1(t) - u_2(t) \| \leqq \| u_{0,1} - u_{0,2} \| + b * \| g_1 - g_2 \| (t), \qquad t \geqq 0 \tag{4.6}$$

where u_i $(i = 1, 2)$ is the generalized solution of (1.1) with $u_0 = u_{0,i}$ $(i = 1, 2)$.

Finally we prove a result concerning an approximation procedure of equation (2.7) which will be used later on.

Let b be completely positive on $[0, T]$ with $b \notin L^\infty(0, T)$. Then it follows from Theorem 3.1 that there exists a unique $k \in L^1(0, T)$ positive decreasing such that

$$k * b = 1 \qquad \text{a.e. on } [0, T].$$

For every $\varepsilon > 0$, define $b_\varepsilon \in \mathrm{AC}[0, \mathrm{T}]$ to be the unique solution to

$$\varepsilon u + k * u = 1 \qquad \text{a.e. on } [0, T];$$

then b_ε is completely positive by Theorem 3.1.

PROPOSITION 4.1. *Let A be a linear m-accretive operator in X and let b_ε be defined by* (4.6). *Let $\lambda > 0$, $u_0 \in X$ and $g \in L^1(0, T, X)$. Then if u_ε is the solution to*

$$u_\varepsilon + b_\varepsilon \cdot A_\lambda u_\varepsilon = u_0 + b * g \qquad a.e.\ on\ [0, T] \tag{4.7}$$

we have

$$\lim_{\varepsilon \to 0} \| u_\varepsilon - u \|_1 = 0 \tag{4.8}$$

where u_ε is the solution to (2.7).

PROOF. Set $v_\varepsilon = u_\varepsilon - u_0$, then v_ε satisfies

$$v_\varepsilon + b_\varepsilon * A_\lambda v_\varepsilon = b_\varepsilon * f \tag{4.9}$$

where $f := g - A_\lambda u_0$, which is equivalent to

$$\begin{aligned} &(\varepsilon v_\varepsilon + k * v_\varepsilon)' + A_\lambda v_\varepsilon = f, \\ &v_\varepsilon(0) = 0, \end{aligned} \tag{4.10}$$

with $\varepsilon v_\varepsilon + k * v_\varepsilon \in W_0^{1,1}(0, T; X)$ $(\varepsilon > 0)$.

We define the following operators in $L^1(0, T; X)$:

$$D(L) := W_0^{1,1}(0, T; X),$$

$$Lu := u' \qquad \text{for } u \in D(L)$$

and

$$D(\Gamma) := \{u \in L^1(0, T; X) : k * u \in W_0^{1,1}(0, T, X)\}$$

$$\Gamma u := (k * u)' \qquad \text{for } u \in D(\Gamma).$$

Moreover define

$$D(L_\varepsilon) := D(L) \cap D(\Gamma)$$

and

$$L_\varepsilon := \varepsilon L + \Gamma, \qquad \varepsilon > 0.$$

Note that $D(L) \subset D(\Gamma)$, hence $D(L_\varepsilon) = W^{1,1}(0, T; X)$ for every $\varepsilon > 0$. From Theorem 3.1 we know that L_ε and Γ are densely defined and m-accretive in $L^1(0, T; X)$. Let $S_{L_\varepsilon}(\cdot)$ (respectively $S_{\varepsilon L}(\cdot)$, $S_\Gamma(\cdot)$) be the C_0-contraction semigroup generated by L_ε (respectively by εL and Γ), then since

$$L\Gamma u = \Gamma L u \qquad \text{for } u \in D(L^2)$$

we have

$$(4.11)\quad S_{L_\varepsilon}(\theta)=S_{\varepsilon L}(\theta)S_\Gamma(\theta)=S_\Gamma(\theta)S_{\varepsilon L}(\theta) \qquad \text{for } \theta\geqq 0 \quad \text{and} \quad \varepsilon>0.$$

Set

$$(Mu)(t):=A_\lambda u(t) \qquad \text{for every } u\in L^1(0,T;X) \quad \text{and} \quad \lambda>0, \text{ a.e. on } [0,T].$$

Clearly M is a bounded m-accretive linear operator in $L^1(0,T,X)$. Moreover if $S_M(\cdot)$ denotes the C^0-contraction semigroup generated by M, one verifies that

$$S_M(\theta)S_{L_\varepsilon}(\theta)=S_{L_\varepsilon}(\theta)S_M(\theta) \qquad \text{for every } \theta\geqq 0, \quad \varepsilon>0.$$

Now, for every $z\in L^1(0,T;X)$ and $\theta\geqq 0$ we have

$$\begin{aligned}\| S_{L_\varepsilon+M}(\theta)z-S_{\Gamma+M}(\theta)z\|_1 &\leqq \| S_{L_\varepsilon}(\theta)z-S_\Gamma(\theta)z\|_1\\ &\leqq \| S_{\varepsilon L}(\theta)z-z\|_1\leqq \| S_L(\varepsilon\theta)z-z\|_1.\end{aligned}$$

Then it follows that $S_{L_\varepsilon+M}(\cdot)z$ converges as $\varepsilon\to 0$ uniformly to $S_{\Gamma+M}(\cdot)z$ for every $z\in L^1(0,T;X)$ on bounded θ-intervals.

Next we show that the type of $S_{L_\varepsilon+M}(\cdot)$ and $S_{L+M}(\cdot)$ is $-\infty$. Since

$$\| S_{L_\varepsilon+M}(\cdot)\| \leqq \| S_{\Gamma+M}(\cdot)\| \leqq \| S_\Gamma(\cdot)\|,$$

it is sufficient to know that the type of $S_\Gamma(\cdot)$ is $-\infty$.

But this follows from Proposition 3.7.

Then given $\omega\in R$ there exists $M\geqq 1$ such that

$$\| S_{\varepsilon L+\Gamma+M}(t)\| \leqq Me^{(\omega-1)t} \qquad \text{for } t\geqq 0$$

and

$$\| S_{\Gamma+M}(t)\| \leqq Me^{(\omega-1)t} \qquad \text{for } t\geqq 0.$$

By using the Trotter–Neveu theorem we obtain that

$$\lim_{\varepsilon\to 0}\|(\omega I+L_\varepsilon+M)^{-1}f-(\omega I+L+M)^{-1}f\|_1=0$$

where f is defined in (4.9), hence

$$\lim_{\varepsilon\to 0}\| u_\varepsilon-u\|_1=0=\lim_{\varepsilon\to 0}\| v_\varepsilon-v\|_1.$$

This completes the proof of Proposition 4.1.

5. Invariance and monotonicity properties of solutions

In this section we shall extend some results obtained in [7] concerning qualitative properties of solutions to (1.1).

PROPOSITION 5.1. *Let A be linear m-accretive in X, b completely positive on* $[0, T]$, $u_0 \in D(A)$, *and* $g \in L^1(0, T; X)$.

Let K_1 be a closed convex subset of X and K_2 be a closed convex cone of X. If

(i) $J_\lambda(K_i) \subseteq K_i$, $i = 1, 2$ *for every* $\lambda > 0$,

(ii) $u_0 \in K_1$ *and* $g(t) \in K_2$ *a.e. on* $[0, T]$.

Then the generalized solution to (1.1) *satisfies*

$$u(t) \in K_1 + K_2 \qquad a.e.\ on\ [0, T].$$

REMARK 5.2. We observe that if in Proposition 5.1 $K_i = P$, $i = 1, 2$ (where P is a closed convex cone of X) the solution to (1.1) satisfies $u(t) \in P$ a.e. on $[0, T]$ provided that x_0, $g(t) \in P$ a.e. on $[0, T]$. Moreover it was proved in [7] that if $h(t) = b(t)x + (b * g)(t)$ with $x \in \overline{D(A)} \cap P$, $g(t) \in P$ a.e. on $[0, T]$ then the solution to $u + b * Au \ni h$ belongs to P a.e. on $[0, T]$. The same is true if $h \in W^{1,1}(0, T; X)$, $h(0) \in \overline{D(A)} \cap P$ and $h'(t) \in P$ a.e. on $[0, T]$.

PROOF. It follows from the definition of the generalized solution that in order to prove the theorem it is sufficient to restrict our attention to the case in which A is replaced by A_λ.

Using (4.6) and the fact that every $f \in L^1(0, T; X)$ with $f(t) \in K_2$ a.e. on $[0, T]$ can be approximated in $L^1(0, T; X)$ by a sequence of continuous functions f_n such that for every $n \in \mathbf{N}$, $f_n(t) \in K_2$ for $t \in [0, T]$, it follows that we can prove the statement with $g \in C(0, T; X)$. Then u_λ (the solution to (2.7)) is the limit in $C(0, T; X)$ of a sequence $(u_\lambda^{(n)})$ defined by

$$\begin{aligned} &u_\lambda^{(0)} = u_0, \\ &u_\lambda^{(n+1)} = r(\lambda^{-1}b) * J_\lambda u^{(n)} + s(\lambda^{-1}b)u_0 + \lambda r(\lambda^{-1}b) * g. \end{aligned} \tag{5.1}$$

Note that $r(\lambda^{-1}b)$, $s(\lambda^{-1}b)$ are nonnegative on $[0, T]$ and $1 * r(\lambda^{-1}b) + s(\lambda^{-1}b) = 1$. It follows that if $K \subseteq X$ is a closed convex set, $x \in K$ and $f \in L^1(0, T; X)$ with $f(t) \in K$ a.e. on $[0, T]$ then

$$s(\lambda^{-1}b)x + r(\lambda^{-1}b) * f \in K \qquad \text{on } [0, T].$$

Indeed by using Remark 3.6 for $t \in (0, T]$ we have

$$
\begin{aligned}
&s(\lambda^{-1}b)x + r(\lambda^{-1}b) * f \\
&\qquad = s(\lambda^{-1}b)x + (1 - s(\lambda^{-1}b))\left[\frac{1}{1 * r(\lambda^{-1}b)} r(\lambda^{-1}b) * f)\right].
\end{aligned}
$$

Since

$$\frac{1}{1 * r(\lambda^{-1}b)}(r(\lambda^{-1}b) * f) \in K$$

the result follows.

Now we proceed by induction on n. Clearly $u_\lambda^{(0)} \in K_1 + K_2$ on $[0, T]$.

We assume $u_\lambda^{(n)}(t) = v_1^{(n)}(t) + v_2^{(n)}(t)$ with $v_1^{(n)}(t) \in K_1$ and $v_2^{(n)}(t) \in K_2$ a.e. on $[0, T]$.

We have

$$u_\lambda^{(n+1)}(t) = v_1^{(n+1)}(t) + v_2^{(n+1)}(t)$$

with

$$v_1^{(n+1)}(t) = (r(\lambda^{-1}b) * J_\lambda^{(n)} v_1^{(n)})(t) + s(\lambda^{-1}b)(t)u_0$$

and

$$v_2^{(n+1)}(t) = (r(\lambda^{-1}b) * J_\lambda^{(n)} v_2^{(n)})(t) + \lambda r(\lambda^{-1}b)(t) * g(t).$$

Then clearly $v_1^{(n+1)}(t) \in K_1$ and $v_2^{(n+1)}(t) \in K_2$ a.e. on $[0, T]$.

This completes the proof of Proposition 3.1.

We shall now consider the monotonicity of generalized solutions of (1.1) in a Banach space with a closed convex cone.

Let (X, P) be a real Banach space and $P \subset X$ a closed convex cone of X. We say that for $x, y \in X$

$$x \leqq y \Leftrightarrow y - x \in P.$$

A function $h \in L^1(0, T; X)$ will be called decreasing if for $s, t \in [0, T] - I$ (here I is a set of measure zero) we have

$$0 \leqq s \leqq t \Leftrightarrow h(t) \leqq h(s).$$

We have the following

THEOREM 5.3. *Let (X, P) be as above and let A be a linear m-accretive operator on the Banach space X.*

Let us assume that

(i) *b is completely positive on* $[0, T]$.

(ii) *For every $\lambda > 0$, $J_\lambda(P) \subset P$.*

(iii) *$g \in L^1(0, T; X)$ is decreasing.*

(iv) *$u_0 \in D(A)$ and $Au_0 \geqq g(t)$ a.e. on* $[0, T]$.
Then the generalized solution to (1.1) *is decreasing on* $[0, T]$.

PROOF. Let $h_0 \in (0, T)$.
Let us define for $0 < h < h_0$

$$g_h(t) = \frac{1}{h} \int_0^h g(t+s)ds \qquad \text{for } t \in [0, T-h_0];$$

then $g_h \in W^{1,1}(0, T-h_0; X)$ and $Au_0 \geqq g_h(t)$ for $t \in [0, T-h_0]$.
Moreover

$$g_h'(t) = \frac{1}{h}(g(t+h) - g(t)) \in P \qquad \text{a.e. on } [0, T-h_0],$$

and

$$\lim_{h \to 0} \| g_h - g \|_{L^1(0,T;X)} = 0.$$

If $b \notin L^\infty(0, T)$ then we replace b by b_ε for $\varepsilon > 0$, defined in (4.6). Hence we may assume that $b \in AC[0, T]$. Next, for $\lambda > 0$ we call u the unique solution to

$$u + b * A_\lambda u = u_0 + b * J_\lambda g_h \tag{5.2}$$

on $[0, T-h_0]$.

Note that $u \in W^{1,1}(0, T-h_0, X)$.

Set $v(t) = u(t) - u_0$ for $t \in [0, T-h_0]$; then v satisfies

$$v + b * A_\lambda v = b * (J_\lambda g_h - A_\lambda u_0). \tag{5.3}$$

Next we differentiate (5.3) and obtain

$$v'(t) + (b * A_\lambda v')(t) = b(t)[J_\lambda g_h(0) - A_\lambda u_0] + (b * J_\lambda g_h')(t). \tag{5.4}$$

Since $J_\lambda g_h(0) - A_\lambda u_0 = J_\lambda(g_h(0) - Au_0) \in -P$ and $g_h'(t) \in -P$ a.e on $[0, T-h_0]$, it follows that (see Remark 5.2) $v'(t) \in -P$, i.e. $v \in C(0, T-h_0, X)$ is decreasing.

Hence u is decreasing on $[0, T-h_0]$.

Next we take the limit as $h \to 0$ in (5.2). We denote by u_h the solution of (5.2) and by w_λ the solution of

$$z + b * A_\lambda z = u_0 + b * J_\lambda g. \tag{5.5}$$

By using Theorem 1 of [5] we have

$$\| u_h(t) - w_\lambda(t) \| \leqq (b * \| J_\lambda(g_h - g) \|)(t) \leqq (b * \| g_h - g \|)(t) \tag{5.6}$$

on $[0, T-h_0]$.

Since $b \in AC(0, T)$ and $g_h \xrightarrow{h\to 0} g$ in $L^1(0, T-h_0; X)$ we have $u_h \xrightarrow{h\to 0} w_\lambda$ in $C(0, T-h_0; X)$.

Then w_λ is decreasing on $[0, T-h_0]$. Since h_0 is arbitrary on $(0, T)$, $w(\cdot)$ is decreasing on $[0, T]$.

In the case where $b \notin L^\infty(0, T)$, the result follows by taking the limit as $\varepsilon \to 0$ and using Proposition 3.6.

Finally, since $J_\lambda g \xrightarrow{\lambda\to 0} g$ in $L^1(0, T; X)$ an application of (4.6) gives

$$\lim_{\lambda\to 0} \| w_\lambda - u \| = 0 \qquad \text{a.e. on } [0, T].$$

Hence the solution of (1.1) is decreasing on $[0, T]$.

REMARK 5.4. If in Theorem 5.3, $u_0 \notin D(A)$ and $A_\lambda u_0 \geqq g(t)$ a.e. on $[0, T]$ for every $\lambda > 0$, then the same conclusion holds.

In what follows we shall apply the results obtained above to the study of the equation

$$u + b * Au = u_0 + b * f(u) \tag{5.6}$$

where b is completely positive on $[0, T]$ for every $T > 0$, A is linear m-accretive in X with $J_\lambda(P) \subset P$ for every $\lambda > 0$, $u_0 \in D(A)$, and $f: X \to X$ satisfies

$$\| f(x) - f(y) \| \leqq M \| x - y \| \qquad \text{for every } x, y \in X \tag{5.7}$$

and some $M \geqq 0$.

First we extend the notion of generalized solution for equation (5.6). As in Crandall–Nohel [9] and Gripenberg [13] the iteration scheme

$$u^{(0)} = -\text{ arbitrary element of } L^1(0, T; X),$$

$$u^{(n+1)} + b * Au^{(n+1)} = u_0 + b * f(u^{(n)}),$$

where $u^{(n+1)}$ is the generalized solution to the linear problem, converges in $L^1_{loc}(0, +\infty; X)$ to some u which we shall call the generalized solution of (5.6). It is also proved there that u is the L^1-limit of u_λ, where u_λ satisfies

$$u_\lambda + b * A_\lambda u_\lambda = u_0 + b * f(u_\lambda). \tag{5.8}$$

As an application of the previous results we have the following

THEOREM 5.5. *Let A, J_λ and b be as in Theorem 5.3. Assume moreover that*

(i) *b is bounded*,

(ii) *there exists* $\omega \geqq 0$ *such that* $f + \omega I$ *is increasing in the sense of order*,

(iii) *there exist* $u_0, v_0 \in D(A)$ *satisfying*

$$u_0 \leqq v_0, \quad Au_0 \leqq f(u_0), \quad Av_0 \geqq f(v_0),$$

Then the generalized solution of

$$\begin{aligned} &u + b * Au = u_0 + b * f(u) \qquad \textit{a.e. on } \mathbf{R} \\ &(\textit{respectively } v + b * Av = v_0 + b * f(v)) \end{aligned} \tag{5.9}$$

is increasing (*respectively decreasing*), *and*

$$u_0 \leqq u(t) \leqq v(t) \leqq v_0 \qquad \textit{a.e. on } \mathbf{R}^+. \tag{5.10}$$

Moreover, for every $w_0 \in \overline{D(A)}$ *such that* $u_0 \leqq w_0 \leqq v_0$, *the corresponding generalized solution* w *satisfies* $u(t) \leqq w(t) \leqq v(t)$, *a.e. on* $\mathbf{R}^+$.

Proof. We prove that u is increasing (the proof of the decreasingness of v is similar) and $u(t) \leqq v_0$.

By considering the scheme

$$\begin{aligned} &u^{(n+1)} + b * (Au^{(n+1)} + \omega u^{(n+1)}) = u_0 + b * (f(u^{(n)}) + \omega u^{(n)}), \\ &u^{(0)} = u_0, \end{aligned} \tag{5.11}$$

it is sufficient to prove that $u^{(n+1)}(\cdot)$ is increasing and $u^{(n+1)}(t) \leqq v_0$ a.e on $\mathbf{R}^+$, when $u^{(n)}$ is increasing and $u^{(n)} \leqq v_0$.

For every $T > 0$ clearly u_0 satisfies the claim; it follows that $u^{(n+1)}$ is increasing.

Indeed (5.11) be rewritten in the form

$$z^{(n+1)} + b * [A + \omega] z^{(n+1)} = z_0 + b * g^{(n)} \tag{5.12}$$

with $z_0 = -u_0$, $z^{(n+1)} = -u^{(n+1)}$, and $g^{(n)}(\cdot) = -(\omega u^{(n)}(\cdot) + f(u^{(n)}(\cdot)))$, and by Theorem 5.3 $u^{(n+1)}(\cdot)$ is increasing and $u^{(n+1)}(t) \leqq v_0$ a.e. on $\mathbf{R}^+$, noting that $Au_0 \leqq f(u_0) + \omega u_0 \leqq f(u_n(t)) + \omega u_n(t)$ a.e. An easy application of (4.6) gives the increasingness of u.

To prove (5.10) we set

$$w_{n+1} = v^{(n+1)} - u^{(n+1)}$$

where $v^{(n+1)}$ is the generalized solution to

$$v^{(n+1)} + b * (Av^{(n+1)} + \omega v^{(n+1)}) = v_0 + b * (f(v^{(n)}) + \omega v^{(n)})$$

and $u^{(n+1)}$ satisfies (5.11).

One sees that w_{n+1} satisfies

$$\begin{aligned}(5.13)\qquad z_{n+1}+b*(Az_{n+1}+\omega z_{n+1})&=v_0-u_0\\ &\quad+b*[(\omega I+f)(v^{(n)})-(\omega I+f)(u^{(n)})]\end{aligned}$$

and then $z_{n+1}\in P$, because $v_0-u_0\in P$ and

$$(\omega I+f)(v^{(n)})-(\omega I+f)(u^{(n)})\in P$$

by induction.

Passing to the limit as $n\to\infty$ in

$$u_0\leqq u^{(n)}(t)\leqq v^{(n)}(t)\leqq v_0 \qquad \text{a.e. on } \mathbf{R}^+$$

one obtains (5.10). The last assertion of the Theorem is proven in a similar way.

As a consequence of the above result we have

THEOREM 5.6. *If in addition to the hypothesis of Theorem* 5.5 *we assume that*

(i) *X is a complete Banach lattice with order continuous norm,*

(ii) $b\notin L^1(0,\infty)$,

then

$$\operatorname*{s-lim}_{t\to\infty} u(t)=u_\infty, \qquad \operatorname*{s-lim}_{t\to\infty} v(t)=v_\infty$$

where u_∞ and v_∞ are respectively the minimal and the maximal solution of the problem $Ax=f(x)$ in the order interval $[u_0,v_0]$.

PROOF. The existence of s-lim $u(t)$ (respectively s-lim $v(t)$) is a direct consequence of the increasingness (respectively decreasingness) of u (respectively v), (5.10) and hypothesis (i).

To prove that u_∞ is the minimal solution of $Ax=f(x)$ in the order interval $[u_0,v_0]$ we proceed as follows.

Let $\omega>0$ and set $g(t):=f(u(t))+\omega u(t)$ a.e. on $\mathbf{R}$. Then equation (5.9) assumes the form

$$(5.14)\qquad u+b*(A+\omega I)u=u_0+b*g$$

by observing that $g\in L^\infty(0,+\infty;X)$ and that there exists $\text{s-lim}_{t\to\infty} g(t)=g_\infty$. An application of Theorem 3.31 of [8] gives

$$Au_\infty = f(u_\infty). \tag{5.15}$$

To conclude the proof one has only to observe that if x is any solution of $Ax = f(x)$ in the ordered interval $[u_0, v_0]$, then $u_0 \leqq u(t) \leqq x$ a.e. on $\mathbf{R}^+$ and then $u_0 \leqq u_\infty \leqq x$. The proof that v_∞ is the maximal solution to $Ax = f(x)$ in $[u_0, v_0]$ is similar.

REMARK 5.7. (i) If $b \notin L^1(0, +\infty)$ and $u_\infty = v_\infty$, then the solution to

$$z + b * Az = z_0 + b * f(z)$$

with $u_0 \leqq z_0 \leqq v_0$ and $Au_0 \leqq f(u_0)$, $Av_0 \geqq f(v_0)$, converges strongly as $n \to \infty$ to the unique solution of $Ax = f(x)$ in the order interval $[u_0, v_0]$.

(ii) If (i) of Theorem 5.6 is replaced by

(i)′ $b \in L^1(0, +\infty)$,

then s-lim $u(t) = u_\infty$ (respectively s-lim $v(t) = v_\infty$) satisfies $Au_\infty \leqq f(u_\infty)$ (respectively $Av_\infty \geqq f(v_\infty)$).

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